

# Reply to Comments by Batic et al.

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We explain why the analysis in our paper [2] is relevant and correct.

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The Comment by [1] on our paper [2] is essentially based on two claims: (i) The integrals used in [2] to determine the QNMs do not exist for negative values of the argument, thereby invalidating the extraction of poles using Born approximation. (ii) The poles of the integral used in [2] to determine the quasinormal modes (QNMs) in the Schwarzschild case come not only from Gamma function but from Whittaker functions as well [eq (7) of [1]].

The second claim is easy to dispose of as erroneous, which we will do first. It is straightforward to see from the formula [9.211.4] of [3] that

$$\int_0^\infty \frac{x^{i\omega/\kappa}}{(x+1)^s} e^{i\omega x/\kappa} = \Gamma\left(1 + \frac{i\omega}{\kappa}\right) \times \Psi\left(1 + \frac{i\omega}{\kappa}, 2 - s + \frac{i\omega}{\kappa}, -\frac{i\omega}{\kappa}\right) \quad (1)$$

where  $\Psi$  is the confluent hypergeometric (Tricomi) function [also written as  $U(a, b, z)$ ]. This function  $\Psi$  is regular for all finite  $a$  and  $b$ . Hence, the only poles arise from the Gamma function and no other functions are needed for determining the pole structure.

Let us now take up the first point (i) which is essentially related to the existence of integrals of the form [e.g., eq (12) of [1]]

$$I = \int_0^\infty dx x^{i\omega/\kappa} e^{i\omega x/\kappa} \quad (2)$$

where  $\kappa = (4M)^{-1}$ . If we introduce the parameters  $\nu = 1 + i\omega/\kappa$  and  $\mu = -i\omega/\kappa$ , the integral can be evaluated as

$$I = \int_0^\infty dx x^{\nu-1} e^{-\mu x} = \frac{\Gamma(\nu)}{\mu^\nu}; \quad \text{Re}(\mu) > 0, \text{Re}(\nu) > 0 \quad (3)$$

The condition  $\text{Re}(\mu) > 0$  translates to  $\omega_I > 0$ , which does not affect our results as we are interested only in

(large) positive values of  $\omega_I$ . If we let  $z = \mu x$ , then we essentially have to evaluate the Gamma function integral

$$\mu^\nu I = I' = \int_0^\infty dz z^{\nu-1} e^{-z} = \Gamma(\nu); \quad (4)$$

The authors claim that this evaluation of the integral as Gamma function is valid only for  $\text{Re}(\nu) > 0$  (which translates to  $\omega_I < \kappa$ ) while we are interested in ref.[2] for large  $\omega_I$ . This objection, too, is easy to take care of.

The point to note is that,  $\Gamma(\nu)$  can be defined by analytic continuation for negative values of  $\nu$ . Even though for  $\text{Re}(\nu) < 0$ , the integrand in Eq.4 behaves as  $\sim z^{\nu-1}$  as  $z \rightarrow 0$  the analytic continuation allows one to define the Gamma function everywhere in complex plane. Once this is done, one can easily extract the poles. A simple, text book way to extract this result is to use the identity:

$$\Gamma(\nu) \Gamma(-\nu) = -\frac{\pi}{\nu \sin \nu\pi} \quad (5)$$

which provides a way to analytically continue  $\Gamma(\nu)$  to negative values of  $\text{Re}(\nu)$ . This also shows that the function has simple poles for negative integral values of  $\text{Re}(\nu)$  arising from the  $\sin(\pi\nu)$  factor, which is also a well-known result. *These are precisely the poles that we are interested in which gives us the desired QNM structure.* Of course, by the very definition of a ‘pole’, the integral diverges at the pole; we stress that the whole exercise is to determine precisely where this occurs! The integral exists in a open neighborhood of the first order pole which is what we used in our analysis. The analytic continuation is based on the standard assumption in scattering theory that the scattering amplitude (and hence the integral in Eq. 2) is analytic everywhere in the  $\omega$  plane, except for a finite number of poles. The key idea developed in [4] and [5] and used in the paper under discussion [2] was to use this assumption, identify the poles and relate it to the QNM.

If one does not want to use the identity in Eq.5 but want to work directly with integral in Eq.4 and extract the information about the poles, that is also possible. We only have to treat the integral in Eq.4 as a limit of a sequence of integrals with a suitable regularization parameter and study the poles. This can be done in many ways and we outline one procedure: Consider the

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integral:

$$I' = \int_0^\infty dz z^{\nu-1} e^{-z} e^{-a/z} = 2a^{\nu/2} K_{-\nu}(2\sqrt{a}) \quad (6)$$

which is well defined even for  $\text{Re}(\nu) < 0$  because of the regulator factor  $e^{(-a/z)}$ . (Here  $K_\nu$  is the modified Bessel function and the relation can be obtained from [3], Sec 8.40-8.43.) We treat the integral in Eq.4 (especially for  $\text{Re}(\nu) < 0$ ) as the limit of  $I'$  when  $a$  is a positive infinitesimal quantity. This gives (again using Sec 8.40-8.43 of [3] and interpreting  $a \rightarrow 0^+$  as a positive infinitesimal value for the regulator):

$$\begin{aligned} I' &= \lim_{a \rightarrow 0^+} \int_0^\infty dz z^{\nu-1} e^{-z} e^{-a/z} \\ &= \lim_{a \rightarrow 0^+} 2a^{\nu/2} K_{-\nu}(2\sqrt{a}) \\ &= \Gamma(\nu) + \lim_{a \rightarrow 0^+} a^\nu [\Gamma(-\nu) + \mathcal{O}(a)] \end{aligned} \quad (7)$$

Note that the last equality in the above equation is valid only when  $\nu$  is *not* a integer. In fact, if we take the limit  $\nu \rightarrow -n$  (where  $n$  is positive integer), the integral diverges *as it should*, because our previous analysis using Eq.5 has already told us that the integral has simple poles at  $\nu = -n$ . We can determine the nature of the singularity arising from these poles trivially. When  $\nu \rightarrow -n$ , we obtain

$$\lim_{\nu \rightarrow -n} (\nu + n)I' = \lim_{\nu \rightarrow -n} (\nu + n)\Gamma(\nu)$$

$$\begin{aligned} &+ \lim_{a \rightarrow 0^+} \lim_{\nu \rightarrow -n} a^\nu [(\nu + n)\Gamma(-\nu) + \mathcal{O}(a)] \\ &= \frac{(-1)^n}{n!} + \lim_{a \rightarrow 0^+} a^{-n} [\dots] \neq 0 \end{aligned} \quad (8)$$

which shows that the singularity of  $I'$  at  $\nu = -n$  is a simple pole for finite regulator. (The procedure is very similar to the  $i\epsilon$  prescription used in field theoretic calculations.) Same arguments are valid for the integral (18) of [1]. We stress that the integrals are not expected to exist for  $\nu = -n$ , which are precisely the poles we want to determine! What we need is a sensible definition of the integral in the open neighborhood of the poles — which can be provided in many ways, of which we have described two.

Finally, we would like to point out that paper in question which is being commented upon [2] is a follow-up of two earlier papers [4] and [5] developing the same technique and containing the same integrals. It is somewhat surprising that the authors of [1] decided to comment a third, follow-up paper rather than the first two! We did not discuss the details of the regularization, analytic continuation etc. in our work [2] as the basic ideas were already implicit in the previous published work.

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